Â The local controlled growth of a perfect Cartwheel type tiling called the quasiperiodic successionÂ

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”The local controlled growth of a perfect Cartwheel type tiling called the quasiperiodic succession”

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Abstract

The modellation of the growth of a decagonal Cartwheel type tiling is not enough forced by the well known matching rules of Penrose tiles. The paper presents a deterministic algorithm which allows the calculation of a perfect Cartwheel type tiling successive out of each cluster cell Q by transferring recursive values to the neighbor cells.

Introduction

Different possibilities to generate a perfect Penrose structure called Cartwheel type tiling are known [1]. The substitution rules generate perfect tilings by the decomposition of a tile into smaller copies of itselve which then can be inflated to the original size of the tile [2]. But it is obvious that they cannot be used to generate local growth.

The well known matching rules can be used for modellation of growth. They act locally but with high probability they do not lead to a perfect Cartwheel type tiling and the growing of other possible patches lead with high probability to configurations which cannot be extended further in accordance to these rules ([3], pp. 84-85).

Extensions of this rules which try to exclude these configurations locally only shift this problem but they do not solve it exactly. A very interesting example is the statistical approach from Ophysen et. al. [4], which is able to generate a Cartwheel type tiling in a limited area. However the matching rules have to be violated and the local principle seems to be extended to a larger environment.

A further approach are the projection methods. But most tilings obtained in this way are not substitution tilings [2] and therefore no Cartwheel type tiling too.

How can be shown with the substitution rules of the Penrose tiles, perfect matching rules have to operate in all scales of substitution. The quasiperiodic succession which is presented here does exactly this.

An elementary aperiodic cluster cell **Q** contains a mechanism which picks up 5 scale values of one neighboring cluster cell, evaluates them and determines 4, 5 or 6 neighbor cell positions with individual scale values each.

This procedure is controlled by 5 constant length $\mathbf{l}^{\mathbf{const}}$ which can be understood as a sliding ruler with two marks which corresponds with the marks of the neighbor cells. The length $\mathbf{l}^{\mathbf{const}}$ is derived as the optimized, averaged length of the self similar, one dimensional, aperiodic interval structures ($\mathbf{Q_a}$, $\mathbf{Q_b}$, $\mathbf{Q_c}$, $\mathbf{Q_d}$, $\mathbf{Q_e}$) from which **Q** is assembled.

The paper presents an arithmetic algorithm on the base of this geometric model which allows to calculate the local structure successive out of each cluster cell without consideration of the global Cartwheel structure. However it can be shown that it corresponds perfectly with this structure which can be generated by substitution.

§1. Conventional construction of Cartwheel type tiling

The various decagonal Penrose tilings all have equivalence relations to each other. The tiling with the best clearness is the Rhombus tiling. It is composed by only two tiles, thick and thin rhombs with angles of 72 degree respectively 36 degree and with consistent edge lengths. In order to avoid periodical configurations it is necessary to define special local matching rules. Therefore usually various marks will be attached to the edges or to the angles of the tiles. Instead of edge marks here is used a black ‘top’- point **T** and a white point **S**, dividing the long diagonal line of the thick rhombus **R** by the golden ratio τ**:1** (where τ**=0.5·(√5+1)**. Alternatively to an arrow from **S** to **T** is a **T**- directioned mark on **S**. Fig. 1a) and 1b) show in which way the tiles can be substituted by tiles scaled down with the factor τ^{-1} in accordance to the matching rules. Fig. 1c) shows a rhombus with a substructure generated by iterated substitutions.

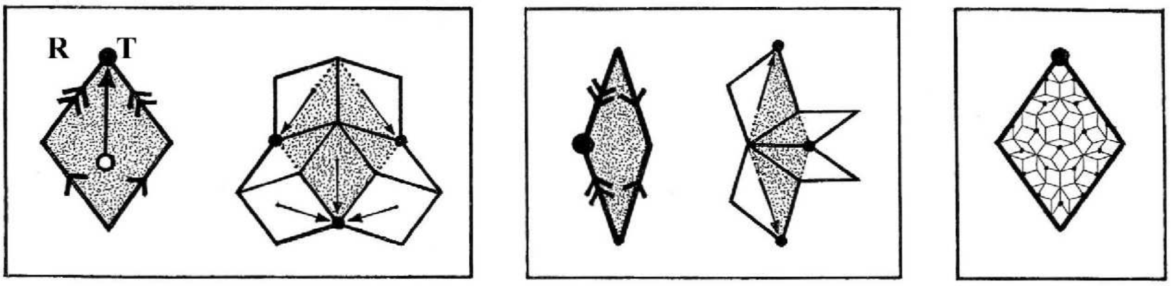


Fig. 1 a) substitution rule of the thick rhombus **R** **b)** substitution rule of the thin rhombus **c)** sub structure of rhombus **R**

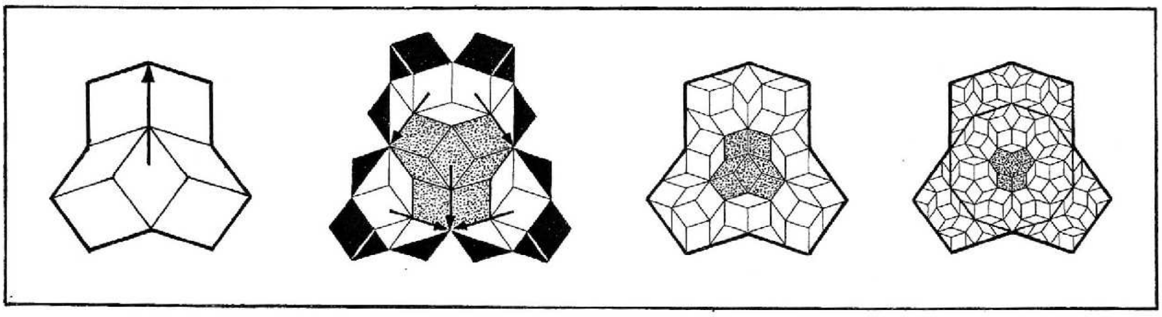


Fig. 1 d) cluster cell, first substitution, second substitution, third substitution

Instead of two various cells the more vivid description of the growth of decagonal quasi crystals is to consider elementary cluster cells which overlap without gaps, see for example [3] and [5] (pp. 63-74).

The simplest cluster cell can be directly derived from the substitution rule of the thick rhombus **R** by an additional definition of two down scaled rhombs which lay outside of the area of the initial rhombus but with defined positions independent of belonging to a thick or a thin neighbor rhombus in the initial scale.

Fig. 1d) shows a cluster cell which can be substituted by fife contractive transformations. Thus the area of the initial cluster cell is filled with the first substructure. The central transformation is only composed by a rotation by 180 degree and a contraction by the factor τ^{-1}. The initial area contains this contracted cluster completely.

The other transformations are composed by a translation, a rotation and a contraction. The parts of the transformed cells outside of the initial cell area are marked black.

The second substructure is generated by the same transformations and contains a cluster cell in the center contracted with the factor τ^{-2} and an equal orientation as the initial cell.

The third substructure on the right contains the central cluster turned around again and contracted with the factor τ^{-3}. This factor becomes very important for the following studies just as the inserted decagon area which is the well known Cartwheel named $\mathbf{C_3}$ (the index specifies the scale of substitution). It can be shown that a Cartwheel generated by 4th or higher scale of substitution (named $\mathbf{C_n}$ with n>3) can be covered by $\mathbf{C_3}$ Cartwheels [1]. Fig. 1e) in the middle shows the decagonal cluster cell **P** elaborated by Petra Gummelt on the base of Rhombus tiling Cartwheel $\mathbf{C_2}$ [3]. The inserted rhombus **R** has an equivalent relation to **P**.

The overlapping rules of **P** base on the same 10 neighbor transformations $\mathbf{h_i}$ as valid for **R** which are shown on the right side of Fig. 1e). To avoid all substituted forms of configurations which cannot be extended infinitely with this rules (examples Fig. 1e)) the matching rules of **R** and **P** would have to be completed by infinite exclusion rules.

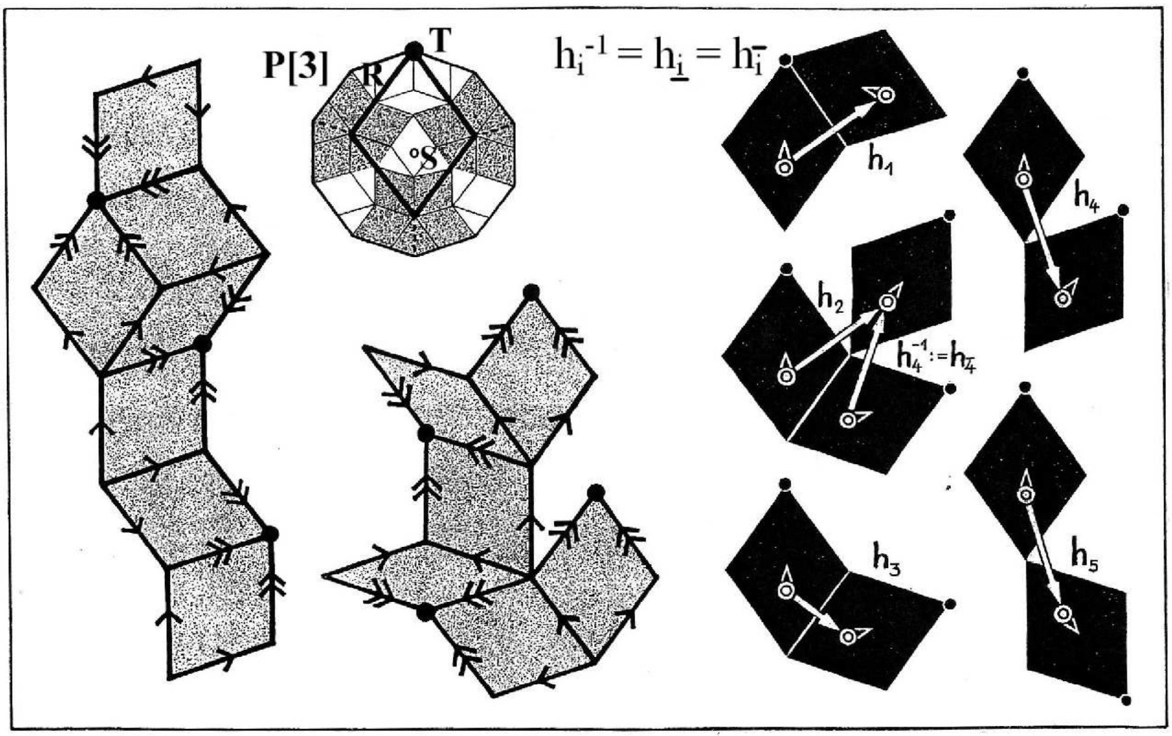
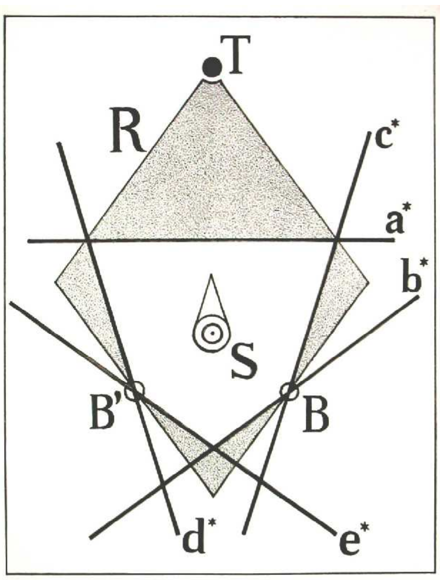


Fig. 1e) top middle: aperiodic decagon [3], left middle: forbidden configurations in a Cartwheel tiling, right: the 5 neighbor transformations and 1 of the inverses (An explanation for the three equivalent notations for the inverse h is given in §4)

§2. Construction methods of Ammann bars, which give the base of the new algorithm of quasiperiodic succession.

Ammann bars which correspond to the Cartwheel type tiling will be called the lattice $\mathbf{L_Q}$. The lattice $\mathbf{L_Q}$ can be generated by adjoining the boundaries $\mathbf{a^*,b^*,c^*,d^*,e^*}$ of a lattice cluster cell **Q** to each thick rhombus **R** of an extended Cartwheel as shown in Fig. 2).



B and **B'** are the bisection points of these boundaries of **R** which do not contain the top point **T**.
The **Q** boundaries $\mathbf{b^*}$ and $\mathbf{c^*}$ intersect the right **R** boundary in its bisecting point **B** with 18 degree each.
The **Q** boundaries $\mathbf{e^*}$ and $\mathbf{d^*}$ intersect the left **R** boundary in its bisecting point **B'** with 18 degree each too.
$\mathbf{a^*}$ is given by the intersection points of c^* and d^* with the upper **T** containing boundaries of **R**.

Fig. 2) equivalence of the **Q** boundaries $\mathbf{a^*,b^*,c^*,d^*,e^*}$ to **R** respectively to the orientated point **S**

An analysis of the Cartwheel type lattice $\mathbf{L_Q}$ shows that the boundaries of $\mathbf{L_Q}$ have only two different distances to one another (in the following called $\mathbf{l^{major}}$ and $\mathbf{l^{minor}}$). Furthermore one can consider that the lattice $\mathbf{L_Q}$ has a strong self similarity to the factor τ^3.
Alternatively $\mathbf{L_Q}$ can be constructed by a composition of fife one dimensional interval structures $\mathbf{Q_a}$, $\mathbf{Q_b}$, $\mathbf{Q_c}$, $\mathbf{Q_d}$, $\mathbf{Q_e}$ (in general called $\mathbf{Q_x}$) which can be generated by a special form of substitution with the factor τ^{-3} which we will call the concordant substitution because it owns a strong self similarity to the factor τ^3 too. The one dimensional interval structures $\mathbf{Q_x}$ have some characteristics which allow to generate them in a local way by an optimized successive process which we will call the quasiperiodic succession.

§3. Perfect matching rules for the one dimensional quasiperiodic interval structure cells $\mathbf{Q_x}$ generated by the one dimensional quasiperiodic succession

Two length $\mathbf{l^{major}}$ and $\mathbf{l^{minor}}$ which have the relation $\mathbf{l^{major}} = \mathbf{l^{minor}} \cdot \tau$ can be substituted by shortened length $\mathbf{l^{major}}/\tau$ and $\mathbf{l^{minor}}/\tau$. Because of the bisection of $\mathbf{l^{major}}/\tau$ such a step of substitution is named discordant. After three discordant steps the initial lengths are substituted by undivided shortened lengths $\mathbf{l^{major}}/\tau^3$ and $\mathbf{l^{minor}}/\tau^3$. Such a step is named a concordant substitution respectively the first pure substructure of the two initial lengths. In the following the τ^{-3} operation will be called simply substructure. An iterated concordant substitution generates the second substructure (Fig. 3, top). An infinite number of iterations generates an infinite structure which allows to calculate an averaged length $\mathbf{l^{const}}$:

In general is valid: $$\mathbf{l^{const.}} = \frac{\mathbf{N(l^{maj})\cdot l^{maj} + N(l^{min})\cdot l^{min}}}{\mathbf{N(l^{maj}) + N(l^{min})}} \qquad (\mathbf{N\ldots}\text{ number of intervals})$$

For infinite numbers $\mathbf{N(l^{maj})}$ and $\mathbf{N(l^{min})}$ generated by substitution we get the ratio:

$$\mathbf{N(l^{maj})} = \tau \cdot \mathbf{N(l^{min})}$$

$$\Rightarrow \mathbf{l^{const.}} = \frac{\mathbf{N(l^{min})}\cdot \tau \cdot \mathbf{l^{min}} \cdot \tau + \mathbf{N(l^{min})\cdot l^{min}}}{\mathbf{N(l^{min})}\cdot \tau + \mathbf{N(l^{min})}} = \mathbf{l^{min}} \cdot \sqrt{5} \cdot \tau^{-1} = 3\cdot \mathbf{l^{major}}/\tau^3 + 1\cdot \mathbf{l^{minor}}/\tau^3$$

These constant lengths can be joined together like the teeth in a comb. If this comb is positioned symmetrically to the aperiodic structure it can be shown that the position of the teeth is always in the middle subinterval of each $\mathbf{l^{major}}$ (middle subinterval is $\mathbf{l^{major}}/\tau^3$.) and of each $\mathbf{l^{minor}}$ (middle subinterval is $\mathbf{l^{minor}}/\tau^3$.)
Vice versa this fact can be used to generate this aperiodic structure by quasi periodic succession on the base of overlap cells. A cell $\mathbf{Q_x}$ is composed out of six subintervals bisected by an interval boundary $\mathbf{x^*}$. Counted up from this boundary to both sides the respective second ($\mathbf{l^{minor}}/\tau^3$) and the third ($\mathbf{l^{major}}/\tau^3$) subinterval together get a unit scale **(0,1)** and will be named $\mathbf{x^{def}}$. The two equal scale values of each cell will be joined by the end markers of $\mathbf{l^{const}}$ which works as a sliding ruler.
In principle two kinds of overlaps are possible. But anytime that one must be selected which makes a correlation between the scale values $\mathbf{x^{def}}$ of the two overlapping cells possible (Fig.3 below). The structure which is generated by the quasiperiodic succession corresponds perfectly with the interval structure which is generated by substitution.

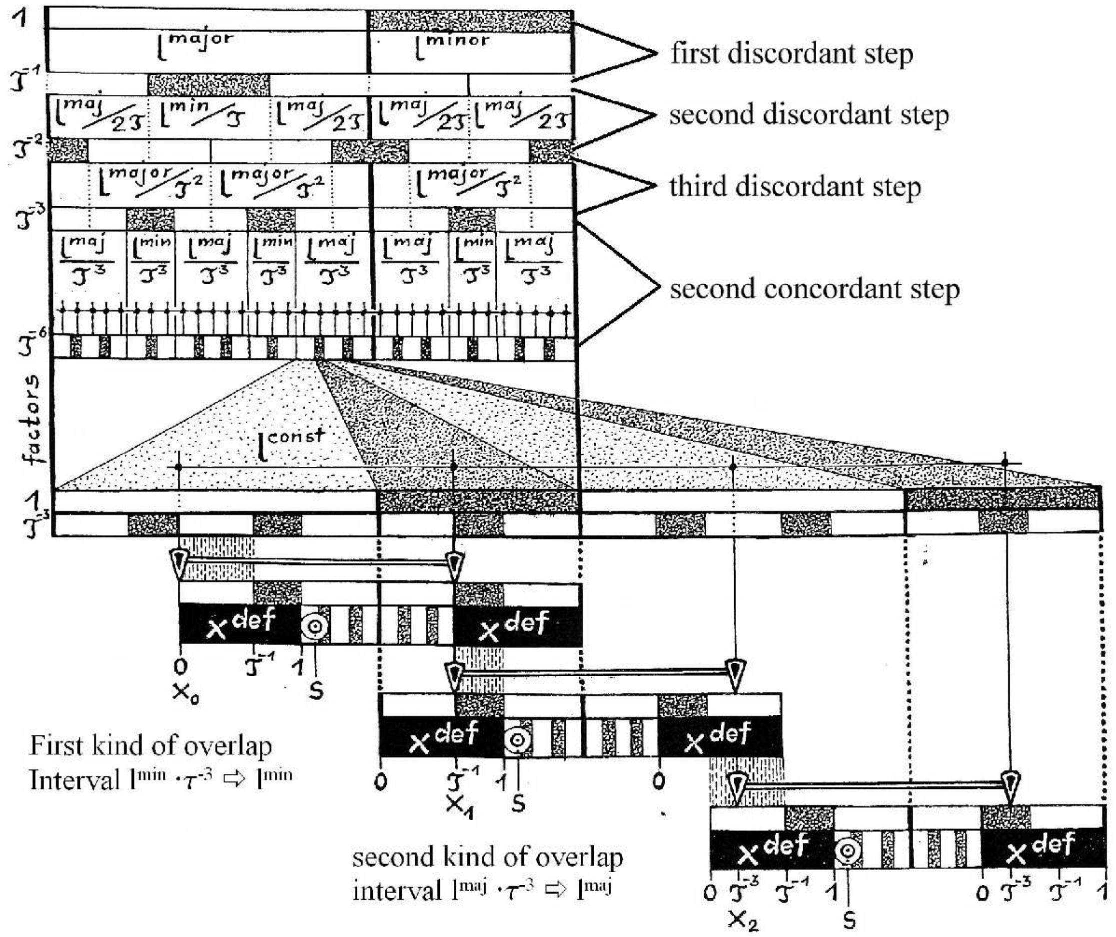


Fig. 3 Equivalence of one dimensional substitution and quasiperiodic succession structure

§4. The construction of the decagonal quasiperiodic lattice cluster Q

In the one dimensional cells $\mathbf{Q_x}$ the bisection of the two zero values is marked by the point **S** whereas the thick interval boundary $\mathbf{x^*}$ bisects the six subintervals of the cell $\mathbf{Q_x}$.

The asymmetrical construction of $\mathbf{Q_x}$ implies that there are 2^5 possibilities to construct a decagonal cell by fife one dimensional cells $\mathbf{Q_x}$ so that all points **S** fall one into another and so the zero values form a decagon.

It can be shown that there is only one reflective symmetrical version with a defined structured trapezoid area that makes a complete overlap structure possible. This version will be called the decagonal quasiperiodic lattice cluster **Q**. The one dimensional structures $\mathbf{Q_x}$ from which the lattice cluster cell **Q** is assembled will be renamed to $\mathbf{Q_a}$, $\mathbf{Q_b}$, $\mathbf{Q_c}$, $\mathbf{Q_d}$, $\mathbf{Q_e}$ anti clockwise (Fig. 4a).

The cluster cell **Q** has an equivalent relation to the rhombus **R** and can be represented by an orientated point **S** like the rhombus **R** too. The equivalence relation allows to apply the neighbour transformations $\mathbf{h_i}$ ($\mathbf{h_{\underline{1}}}$, $\mathbf{h_{\underline{4}}}$, $\mathbf{h_{\underline{3}}}$, $\mathbf{h_{\underline{2}}}$, $\mathbf{h_{\underline{5}}}$, $\mathbf{h_5}$, $\mathbf{h_4}$, $\mathbf{h_3}$, $\mathbf{h_2}$, $\mathbf{h_1}$) to the cluster cell **Q**. The underlined numbers mark inverse transformations and allow a shortened index notation for sequences (example: $\mathbf{h_4^{-1}(h_2^{-1}(R_0)) = R_{0\underline{2}\underline{4}}}$). The underlined notation in the text is equivalent to the toplined numbers in the handmade graphics and the recursive formula set.

The shortening of the $\mathbf{b^{def}}$ and $\mathbf{e^{def}}$ intervals is forced by the equivalence to the matching rules of the rhombus **R**. Nevertheless the intervals of all (fife) cells $\mathbf{Q_x}$ will be named $\mathbf{x^{def}}$ in general statements.

Fig. 4b) shows the neighbour transformation $\mathbf{h_4}$ applied to the rhombus $\mathbf{R_0}$. Like each transformation $\mathbf{h_i}$, $\mathbf{h_4}$ is compound by a translation and a rotation about the point **S**. The

right side of Fig 4b) shows in which way the interval $\mathbf{a}^{def}$ of the cell $\mathbf{Q_0}$ (represented by $\mathbf{S_0}$) correlates with the interval $\mathbf{b}^{def}$ of the cell $\mathbf{Q_{04}}$ (represented by $\mathbf{S_{04} = h_4 (S_0)}$).
In general $\mathbf{b(h_4(Q))=\tau^{-1} - a (Q)}$ is valid. That means that the transformation $\mathbf{h_4}$ is not allowed for a cell $\mathbf{Q}$ with a value $\mathbf{a > \tau^{-1}}$. The values $\mathbf{a_0, b_0, c_0, d_0, e_0}$ of $\mathbf{Q_0}$ will be fixed on $\delta=10^{-8}$ to avoid values on the sub boundaries which limit the $\mathbf{x}^{def}$ intervals.

$$\Rightarrow \mathbf{b_{04} = \tau^{-1} - a_0 = \tau^{-1} - \delta} \quad \Rightarrow \quad \mathbf{b_{04}\ \varepsilon\ b^{def}}, \qquad \mathbf{b^{def} = \{\ b \mid 0 < b < \tau^{-1}\ \}}$$

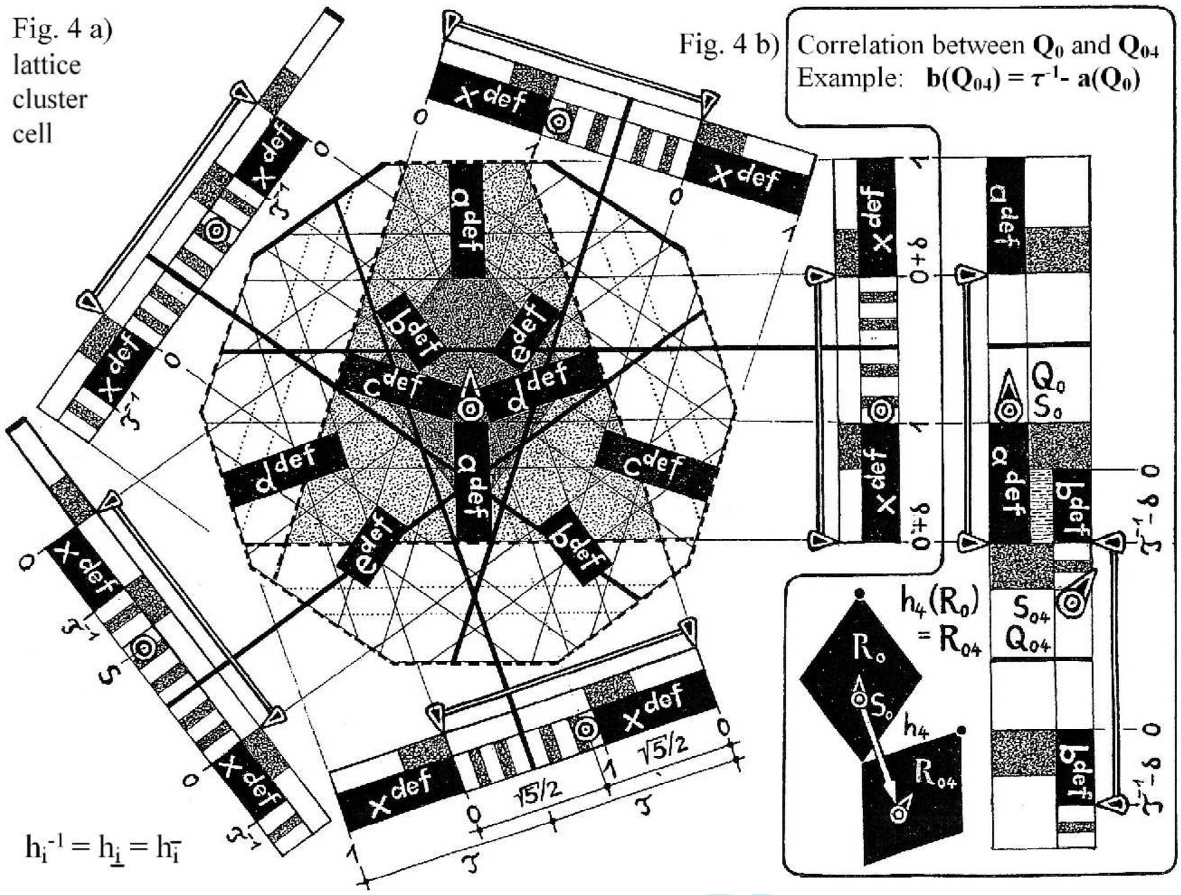


Fig. 4 Lattice cluster cell $\mathbf{Q}$ with ruler position of $\mathbf{Q_0}$ assembled out of 1 dim. cluster cells $\mathbf{Q_a}$, $\mathbf{Q_b}$, $\mathbf{Q_c}$, $\mathbf{Q_d}$, $\mathbf{Q_e}$

§5. The quasiperiodic succession of the lattice cluster cells Q started up from $\mathbf{Q_0}$ respectively from equivalent $\mathbf{R_0}$

In the same way as the correlation of the example $\mathbf{b(Q_{04})} = \mathbf{\tau^{-1} - a(Q_0)}$ leads to $\mathbf{b_{04}\ \varepsilon\ \{b^{def}\}}$, the values $\mathbf{c_{04}}$, $\mathbf{d_{04}}$, $\mathbf{e_{04}}$ and $\mathbf{a_{04}}$ correlate to $\mathbf{b_0, c_0, d_0}$ and $\mathbf{e_0}$ and lead to $\mathbf{c_{04}\ \varepsilon\ \{c^{def}\}, d_{04}\ \varepsilon\ \{d^{def}\}, e_{04}\ \varepsilon\ \{e^{def}\}, a_{04}\ \varepsilon\ \{a^{def}\}}$. Because of this complete set of correlations $\mathbf{Q_{04}}$ will be called the quasiperiodic successor of $\mathbf{Q_0}$.
Fig. 5a) shows the set of fife generally correlating $\mathbf{x}^{def}$ values between a cell $\mathbf{Q}$ and a cell $\mathbf{h_{\underline{3}}(Q)}$.
It is obvious that $\mathbf{Q_{0\underline{3}}}$ cannot be successor of $\mathbf{Q_0}$ because it's a-value $\mathbf{a_0=0 + \delta}$ has no correlation to the parallel $\mathbf{d_{0\underline{3}}}$ value of $\mathbf{h_{\underline{3}}(Q_0)}$ (compare Fig. 5a)).
But the values of the cell $\mathbf{Q_{0\underline{2}}}$ which has reflective symmetry to the cell $\mathbf{Q_{04}}$ relative to the symmetry line of $\mathbf{Q_0}$ all have a correlation to the values of $\mathbf{h_{\underline{3}}(Q_{0\underline{2}})}$ so that $\mathbf{Q_{0\underline{2}\underline{3}}}$ will be quasiperiodic successor of $\mathbf{Q_{0\underline{2}}}$ (compare Fig. 5a) with Fig. 5b)).

Using only the conventional matching rules further $\mathbf{Q_{0\underline{233}}}$ would be allowed. But the perfect matching rules of the quasiperiodic succession forbids this transformation because of the **a**-value $\mathbf{a_{0\underline{233}} = 1 + \tau^{-3} - \delta}$ and so $\mathbf{a_{0\underline{233}}}$ is not ε $\{\mathbf{a^{def}}\}$ with $\mathbf{a^{def} = \{ a \mid 0 < a < 1 \}}$

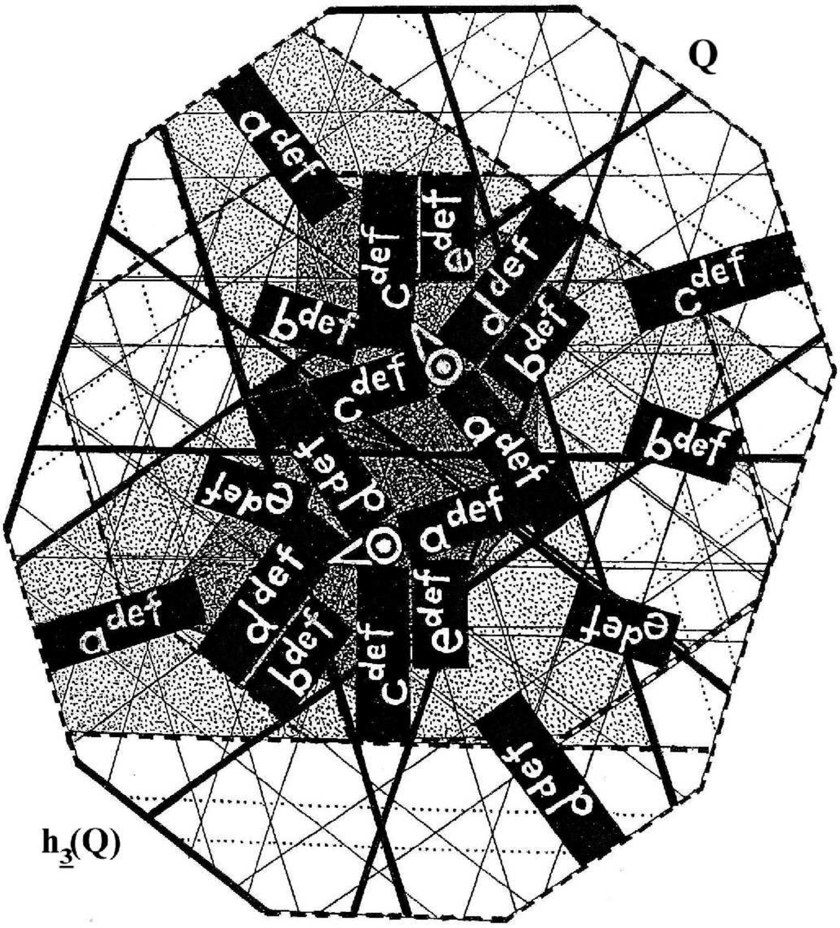


Fig. 5a) Transformation $\mathbf{h_{\underline{3}}}$ of two overlapping cells **Q,** with correlation between parallel scale value bars

(Fig. 5b) top left) shows the proof of the transformation $\mathbf{h_{\underline{1}}(R_{0\underline{13}})=R_{0\underline{131}}}$.
The drawing demonstrates that the end marks of the sliding rulers $\mathbf{l_a^{const}, \ldots, l_e^{const}}$ all are positioned in the intervals $\mathbf{a^{def}, \ldots, e^{def}}$.
This denotes that $\mathbf{R_{0\underline{131}}}$ is a quasiperiodic successor of $\mathbf{R_{0\underline{13}}}$ in accordance to the Cartwheel tiling too.

Theorem:
"The accordance to the Cartwheel tiling is generally given for all transformations which are generated by the quasiperiodic succession starting from $\mathbf{Q_0(R_0)}$***."***

Each proof whether a transformation $\mathbf{h_j}$ is a quasiperiodic successor of a cell **Q** or not can be verified by fife equations which specify the correlations between the scale values **a, b, c, d, e** of the neighboured cells **Q** and $\mathbf{h_j(Q)}$.
A recursive formula which contains the ten possible neighbour transformations $\mathbf{h_j}$ with **j** ε $\{\underline{1},\underline{4},\underline{3},\underline{2},\underline{5},5,4,3,2,1\}$ always give one of nine neighbor types which are postulated by Petra Gummelt [3] (pp. 76-77) for a perfect Penrose structure.

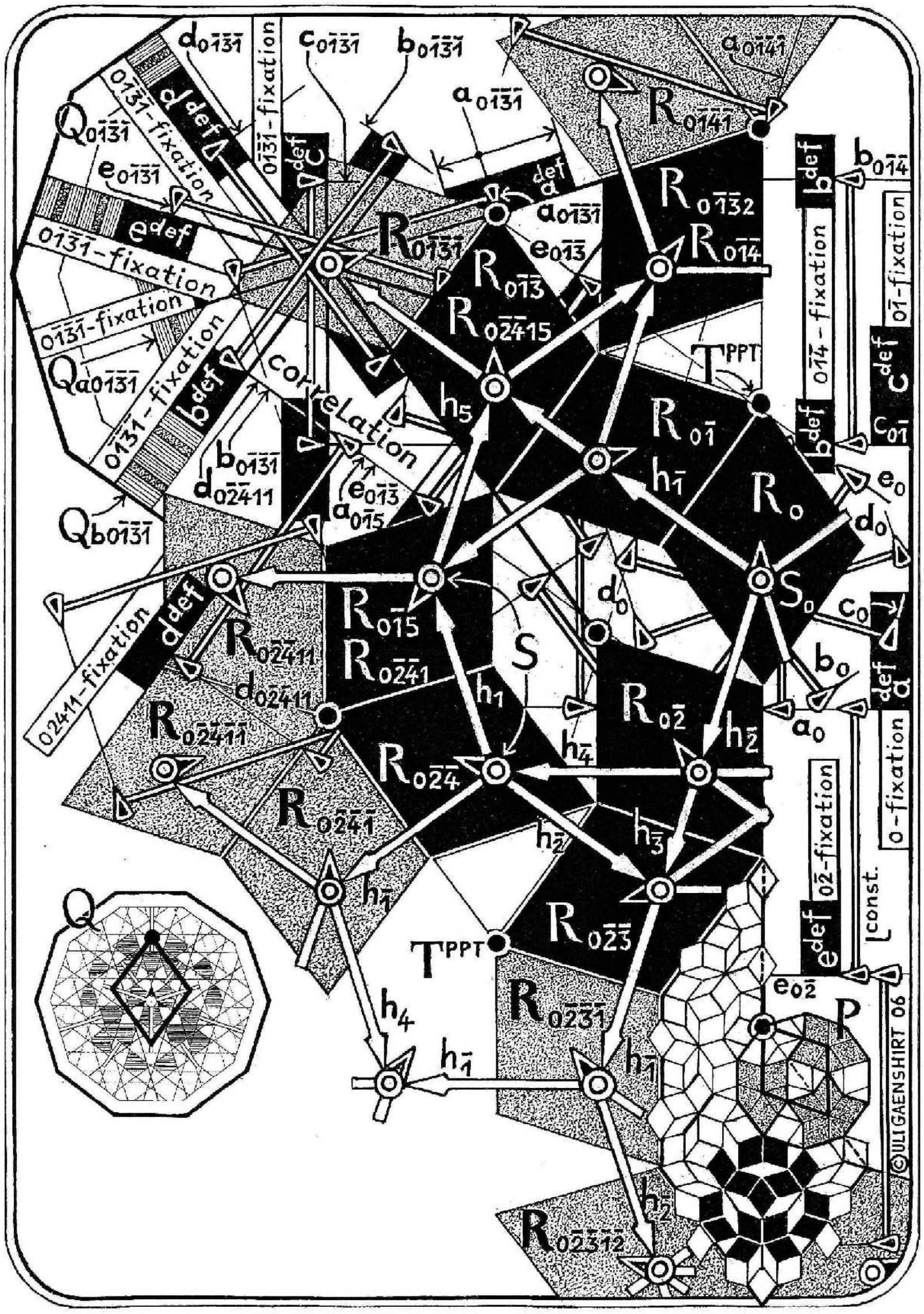


Fig. 5b) quasiperiodic succession applied to rhombus tiling

§6. The recursive formula set for generation of the Cartwheel tiling

The start values $\mathbf{a}^{id} = \mathbf{a}_0 = \delta, \ldots, \mathbf{e}^{id} = \mathbf{e}_0 = \delta$ have to be inserted in the upper set of recursive formula and in its 10 proofs (Fig. 6). The proofs $\mathbf{h}_{\underline{1}}(\mathbf{Q}_0)$, $\mathbf{h}_{\underline{2}}(\mathbf{Q}_0)$, $\mathbf{h}_4(\mathbf{Q}_0)$ and $\mathbf{h}_1(\mathbf{Q}_0)$ give the truth value **t** = **T**rue because the equation sets of these proofs have fife solutions which belong to the respective $\mathbf{x}^{def}$ values **(**ε **{**$\mathbf{x}^{def}$**}).**

The truth tabulation gives for $\mathbf{Q}_0$ the neighbour type 1. The solution values of these four proofs will become the **id** values of four new **id** cards and will be inserted in their proofs too. For $\mathbf{Q}_{0\underline{1}}$ follows type 5, for $\mathbf{Q}_{0\underline{2}}$ type 8, for $\mathbf{Q}_{04}$ type 7 and for $\mathbf{Q}_{01}$ follows type 3. Now 18 new **id** cards can be calculated whereas some of them have the same identity.

The iterated continuation of this process generates a perfect Cartwheel structure independent of an application to **Q, R, P** or other cluster cells with an equivalence relation to **Q**.

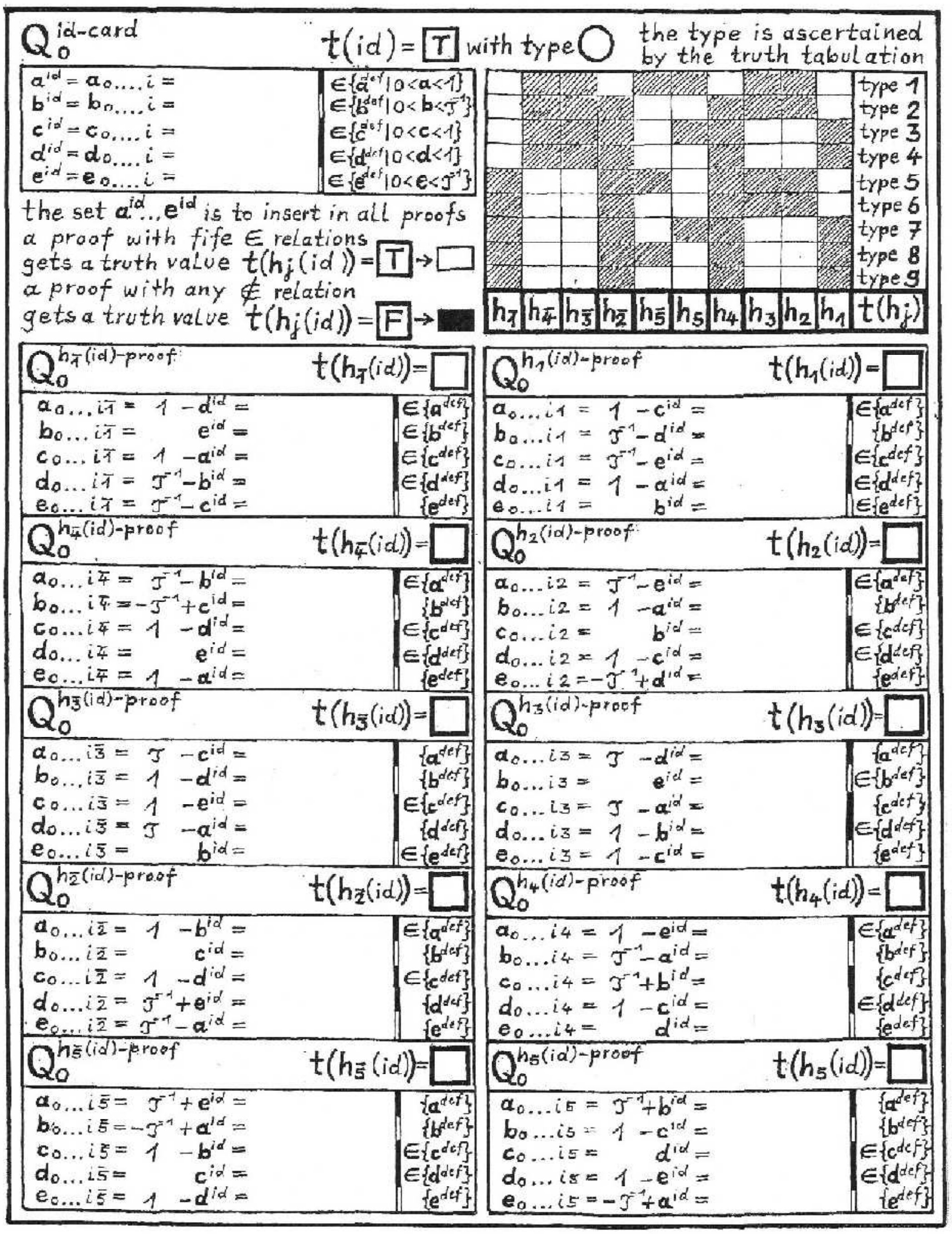


Fig. 6 The recursive formula set for generation of the Cartwheel tiling

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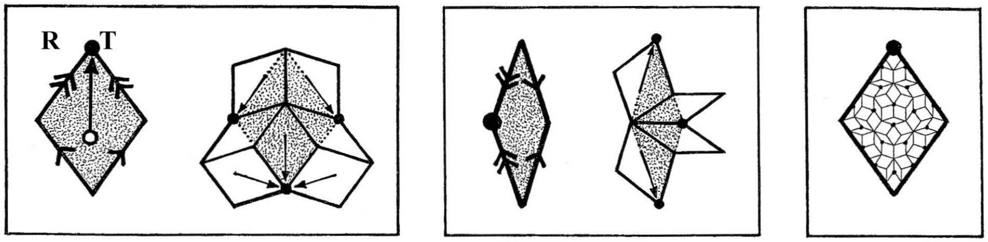

Figure 1a-c
163x42mm (300 x 300 DPI)

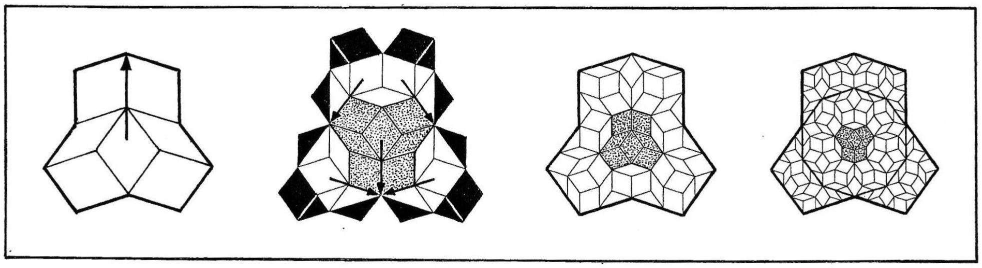

Figure 1d
163x46mm (300 x 300 DPI)

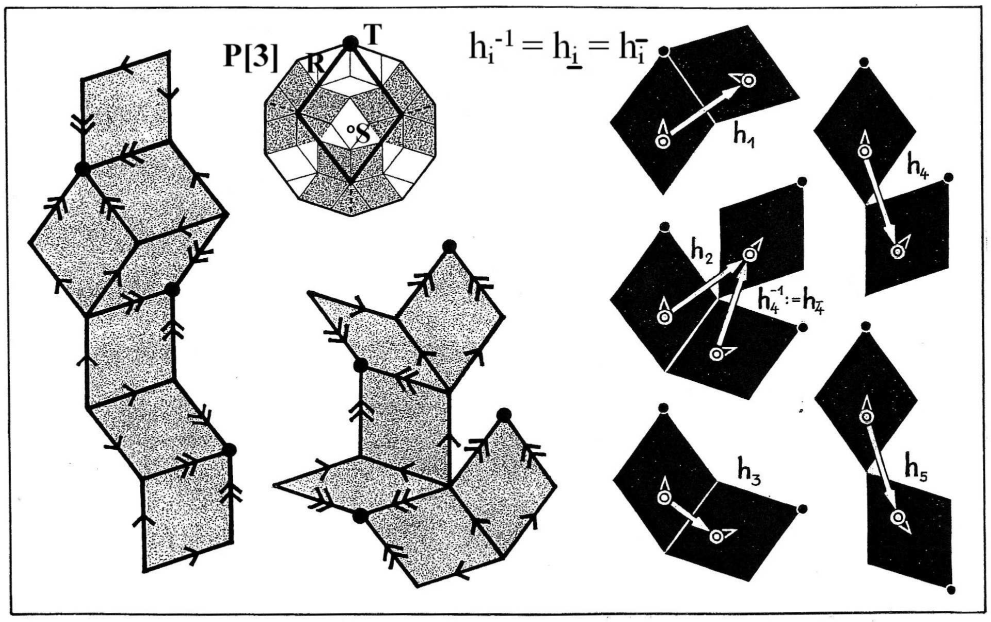

Figure 1e
164x103mm (300 x 300 DPI)

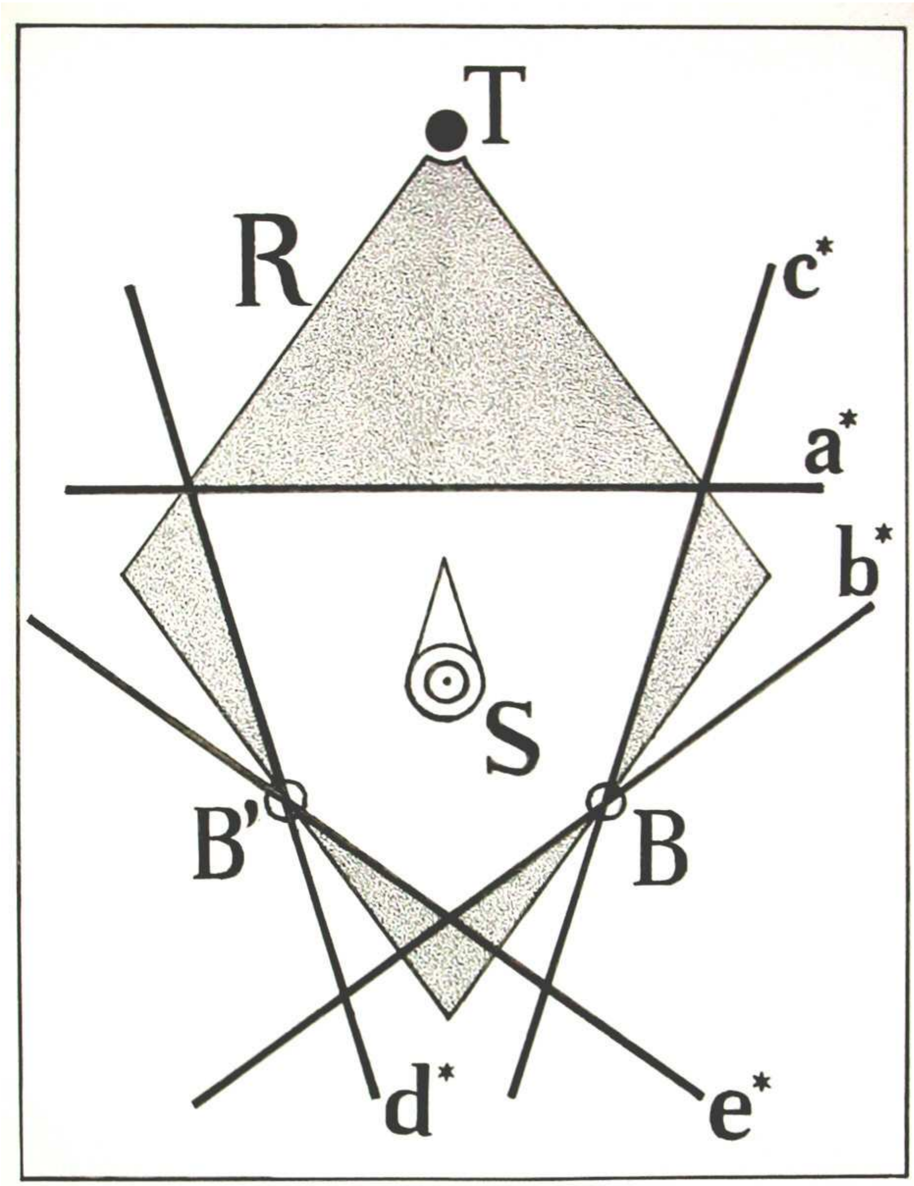

Figure 2
158x207mm (150 x 150 DPI)

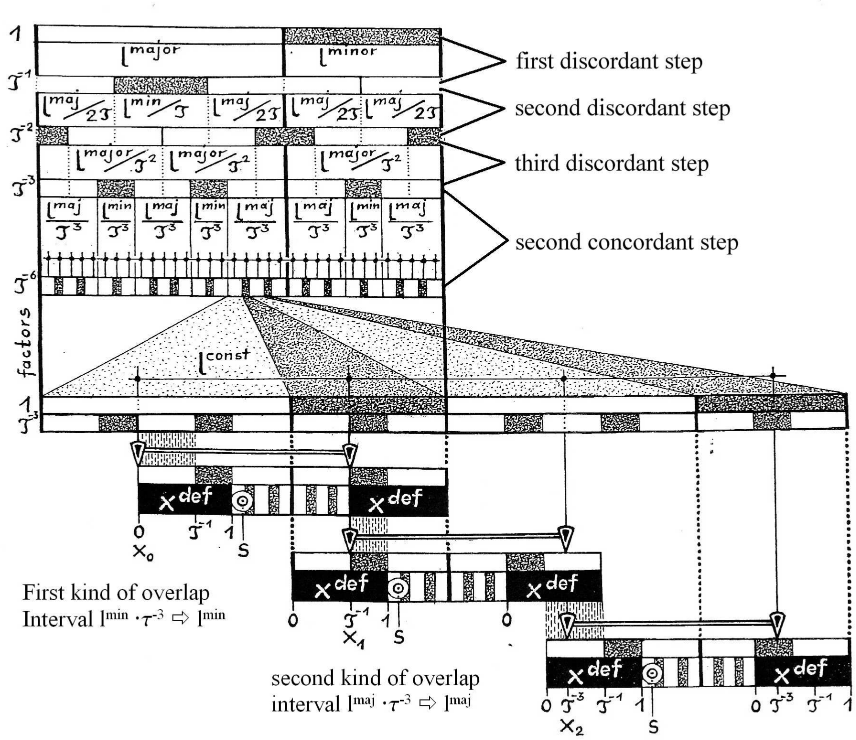

Figure 3
177x133mm (300 x 300 DPI)

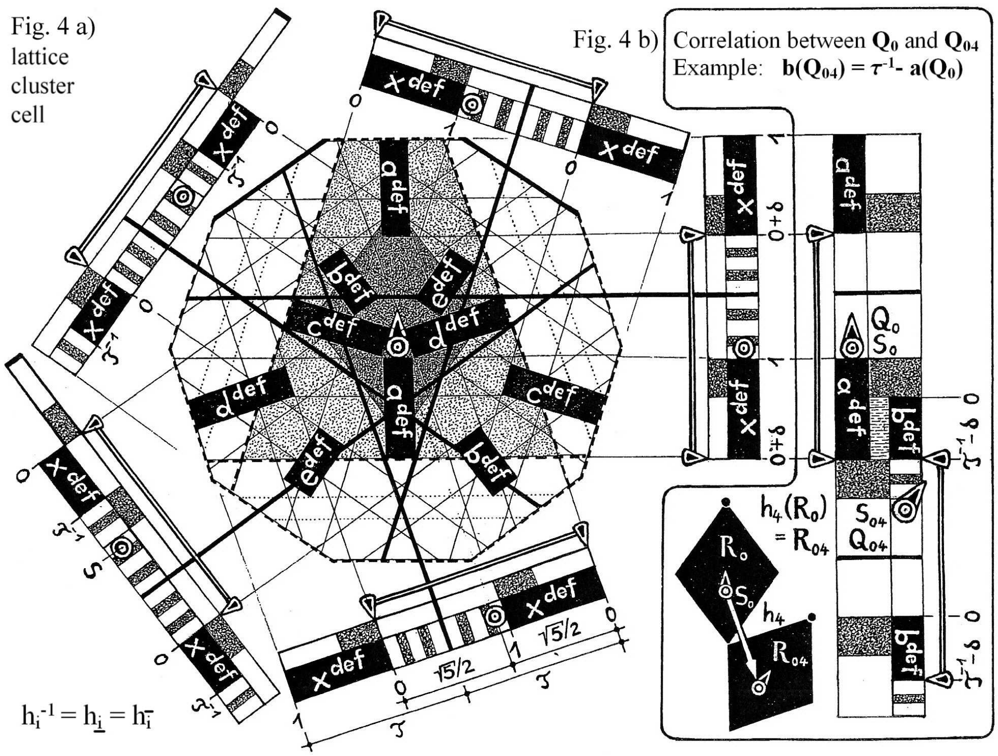

Figure 4
166x126mm (300 x 300 DPI)

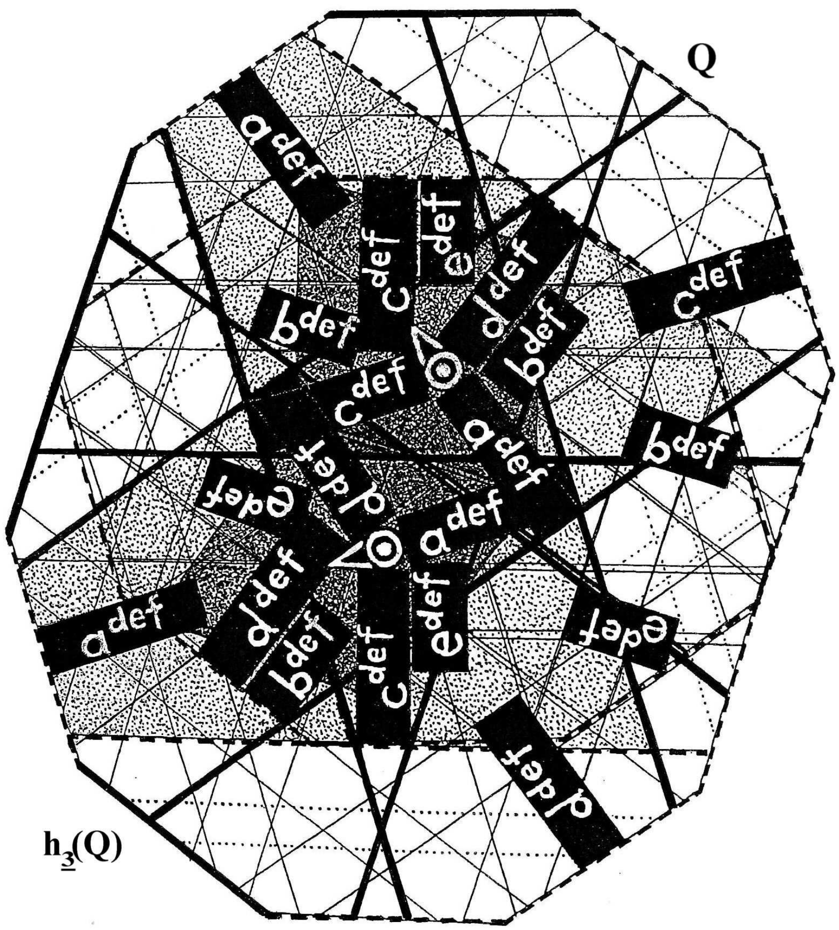

Figure 5a
156x145mm (300 x 300 DPI)

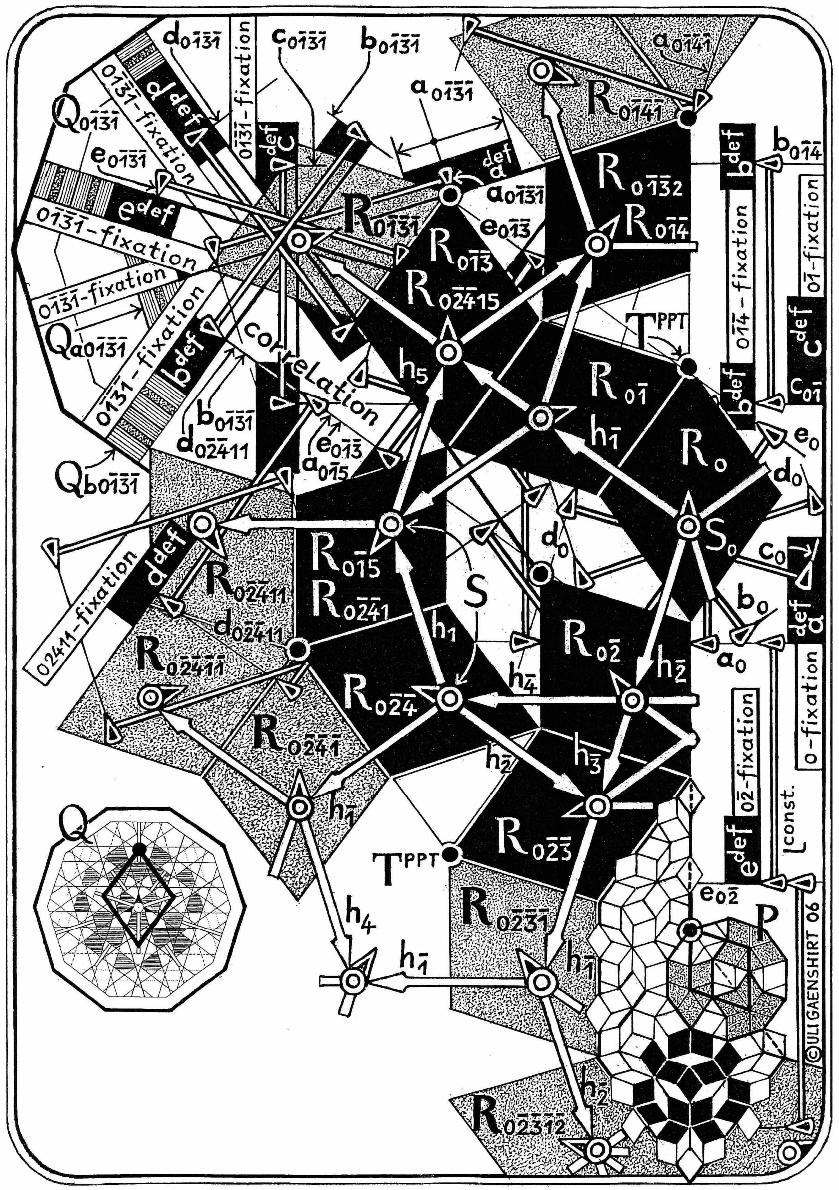

Figure 5b
208x295mm (300 x 300 DPI)

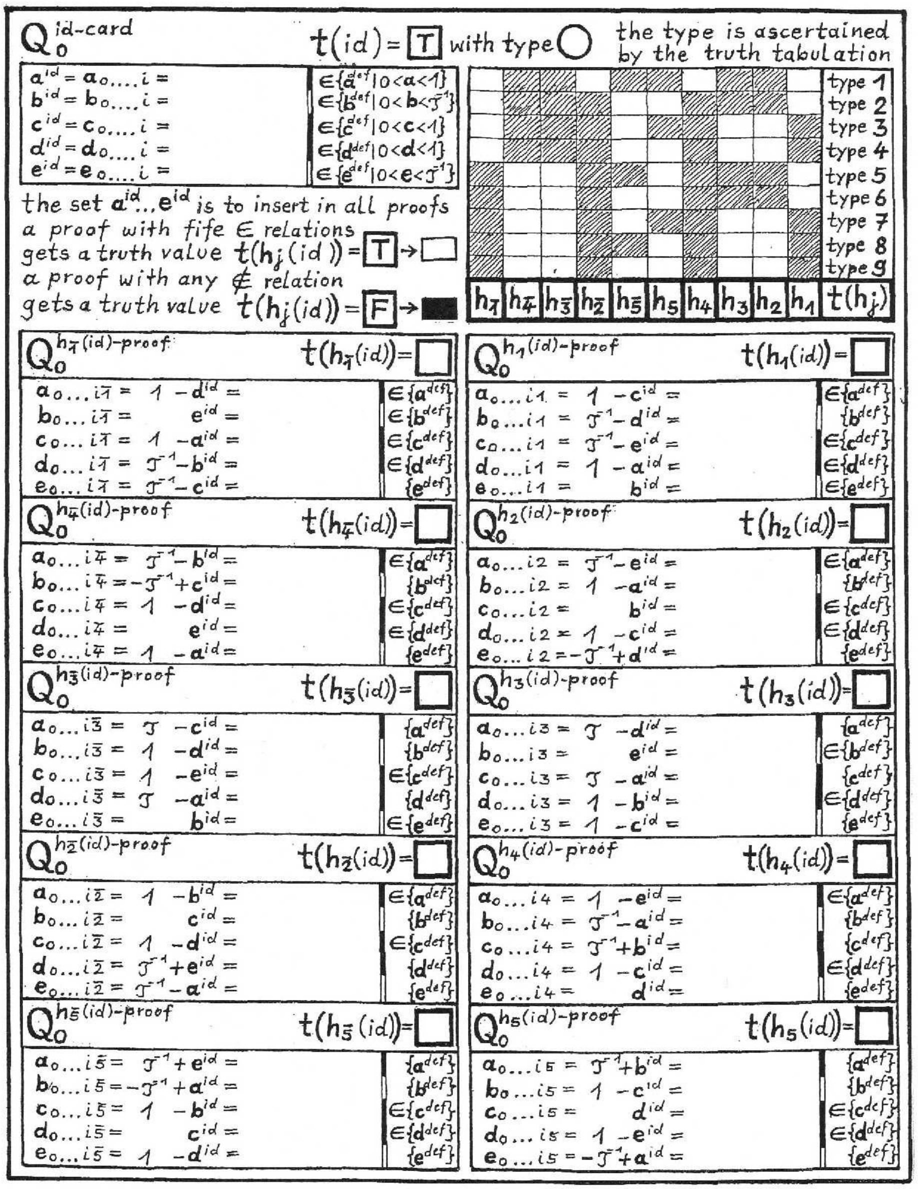

Figure 6
159x206mm (200 x 200 DPI)